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Semantic Alignment of Fractions and Decimals with Discrete Versus Continuous Entities: A Cross-national Comparison

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Abstract

Previous work has shown that adults in the United States selectively use fractions and decimals to model discrete and continuous entities, respectively. However, it is unclear whether this apparent semantic alignment between the format of rational numbers and quantitative ontology is specific to the American education system, the English language, or measuring conventions (primarily imperial measures). Here we test whether similar alignments hold for Korean college students who differ from American students in educational background, language, and measurement conventions. Across three experiments, we found that the alignments found in the United States were generally replicated in South Korea. Relative to Americans, Korean students showed an overall bias towards using continuous representations, perhaps related to their exclusive use of the metric measurement system and to differences in instructional practice identified in a textbook analysis.

Keywords: Cross-national comparison, semantic alignment, continuous and discrete quantities, fractions, decimals

Introduction

People's interpretation and use of arithmetic operations is guided by *semantic alignment* between mathematical and real-life situations. The entities in a problem situation evoke semantic relations (e.g., tulips and vases evoke the functionally asymmetric "contain" relation), which people align with analogous mathematical relations (e.g., the non-commutative division operation, tulips/vases) (Bassok, Chase, & Martin, 1998; Guthormsen et al., in press). Rapp, Bassok, DeWolf and Holyoak (2015) found that a form of semantic alignment guides the use of different formats for rational numbers—fractions and decimals. Specifically, adults in the United States selectively use fractions and decimals to model discrete (i.e., countable) and continuous entities, respectively. DeWolf, Bassok and Holyoak (2015) also demonstrated that American college students prefer to use fractions to represent ratio relations between countable sets, and decimals to represent ratio relations between continuous quantities.

The apparent semantic alignment between fractions and decimals with discrete and continuous quantities, respectively, potentially reflects a basic ontological distinction between quantity types (Bassok & Olseth, 1995). However, this alignment has so far been demonstrated only with American students. Given that the distinction between discrete and continuous entities has linguistic and cultural correlates (Geary, 1995), it is possible that non-English-speaking students from a different culture would not align distinct mathematical symbols with distinct types of quantity.

Here we report a cross-national investigation of whether alignments between rational numbers and discrete and continuous entities are found for students in South Korea. South Korea provides a particularly interesting comparison to the U.S. because the language is structurally different from English, and the culture and education system differ with respect to several factors that may impact students' conceptions of rational numbers. First, in comparison to the U.S., South Korea has excelled in mathematics achievement in recent years. According to the 2012 PISA results (OECD, 2012), South Korea ranked 5th in mathematics achievement (compared to the 36th-place standing of the U.S.). There is evidence that much of this superior achievement in Asian countries can be explained by educational techniques that emphasize achieving deeper conceptual understanding and mastery before moving on to more complex concepts (Perry, 2000; Stigler, Fernandez & Yoshia, 1996; Bailey et al., 2015). The number-naming system is more systematic in Korean than English. Also, base-10 units (metric units) are used exclusively in South Korea, whereas non-base-10 (e.g., imperial) units are widely used in the U.S., and are known to affect students' interpretation and use of fractions and decimals (Rapp et al., 2015).

Experiment 1

The purpose of Experiment 1 was to test whether adults in South Korea show the same pattern of alignment between rational numbers and continuous and countable entities, as found in the U.S. To this end, we asked Korean undergraduate students to generate word problems that contained either fractions or decimals, and examined the entities (continuous vs. countable) they described in their generated problems.

Participants

A total of 71 undergraduate students (male = 25; mean age = 21.39) from Yonsei University in South Korea participated in the study for course credit. A randomly-selected half of these participants generated decimal word problems and the other half generated fraction word problems.

Design, Materials, and Procedures

The study was a between-subjects design with one factor: number type (fraction vs. decimal). Experimental materials were adapted from Experiment 1 of Rapp et al. (2015). Translations of the English versions were created by two Korean-English bilingual researchers, and then back-translated into English to ensure accuracy (also in Experiments 2-3). Participants were given a single sheet of paper with three examples of simple word problems provided at the top. Three examples involved one countable entity (30 marbles), one continuous entity (5 kilometers), and one discretized mass entity (four 2-kilogram bags of sugar). The unit "kilogram" was used in place of "pound". All of these examples were presented with whole numbers. Participants were then asked to generate two word problems with their own numbers. Depending on the condition, they were told that numbers in their problems had to be fractions (e.g., 1/4, $1^1/_2$, 5/2), or decimals (e.g., 0.25, 1.5, 2.5). Participants completed the study using paper and pencil. There was no time limit.

Results

There were a total of 142 problems constructed (70 decimals, 72 fractions). The constructed problems were coded using the classification scheme developed by Rapp et al. (2015). Problems were classified as fraction or decimal based on the number type that appeared in the problem text. Problems were classified as continuous or countable (i.e., discrete) based on the entities that appeared in the constructed problems. Continuous problems involved entities that are referred to linguistically as "mass nouns" (e.g., those varying continuously in weight, volume, or length), whereas countable problems involved either discrete or explicitly discretized entities. Discrete entities were sets of individual objects that cannot be broken down into natural equal units (e.g., marbles, balloons, or grapes), and discretized units were continuous entities that were parsed into equal countable parts (e.g., an apple cut into equal slices, or a rectangle divided into equal squares).

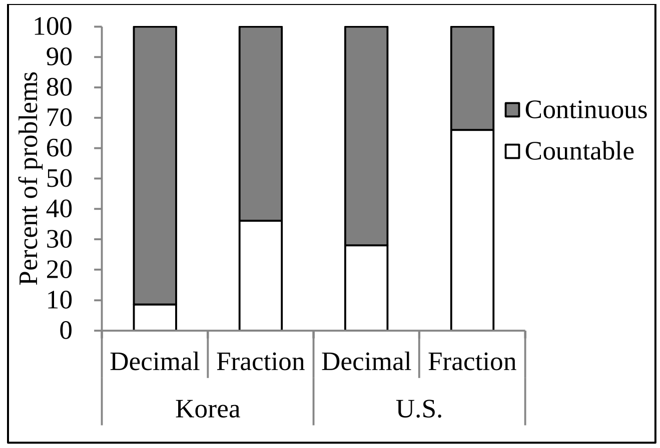

Figure 1. Distribution of countable and continuous problems in decimal and fraction problems for students in South Korea (left panel) and the U.S. (right panel). The U.S. results are from Rapp et al. (2015, Ex. 1).

The left panel of Figure 1 shows the distribution of countable and continuous problems in the decimal and fraction problems. Overall, students generated more continuous problems with decimals than fractions. A chi-square test confirmed that number type (decimal vs. fraction) and continuity (continuous vs. countable) were significantly associated, $\chi^2(1) = 15.42$, $p < .001$. For comparison, the right panel shows the results from U.S. undergraduates (Experiment 1 of Rapp et al., 2015). Overall, there was a consistent pattern of alignment across the two nations in that students tend to use decimals to represent continuous entities and fractions to represent discrete or countable entities. However, Korean students showed an overall bias towards using continuous rather than countable quantities. In fact, unlike American students, Korean students used continuous quantities more often than countable quantities when creating fraction word problems.

Experiment 2

Experiment 2 tested the alignment of number type with quantity by asking students to choose either a continuous or a discrete depiction of fractions and decimals, which were paired with continuous or discrete entities.

Participants

A total of 57 undergraduate students (male = 14; mean age = 21.12) from Yonsei University participated in the study for course credit.

Design, Materials, and Procedures

The study was a 2 (number type: fraction vs. decimal) X 2 (entity type: continuous vs. countable) within-subjects design. There were two trials of each experimental condition, resulting in a total of eight trials per participant. Experimental materials were constructed by adapting the

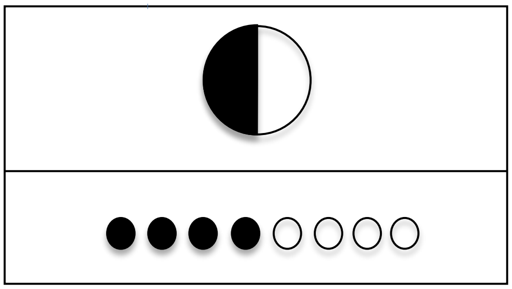

Figure 2. Options provided to represent continuous (circle) and discrete (dots) representations in Experiment 2. In the experimental material, the representations were labeled as Type 1 and Type 2, respectively, and presented vertically as shown.

materials used in Experiment 2 of Rapp et al. (2015). Because imperial units (pound, mile) are seldom used in Korea, these were replaced with metric units (liter, degree in Celsius).

Each participant saw eight different expressions, each including either a fraction or a decimal and either a countable (pen, sandwich, book, banana) or continuous (kilometer, liter, degree in Celsius, kilogram) entity type. Four fractions were used (3/4, 5/8, 4/9, 2/7), and their magnitude-equivalent decimals (.75, .63, .44, .29). For example, a participant might see “3/4 kilometer” or “.75 sandwich.” Assignments of entity type and number type were counterbalanced so that half of the participants received a fraction with a particular entity (e.g., 3/4 sandwich) and half received the equivalent decimal with that same entity (e.g., .75 sandwich). Thus, each participant saw eight of the 16 possible pairings of number and entity type.

The dependent variable was whether participants selected a continuous circle representation or a discrete dots representation for the number type-entity type expressions (see Figure 2). Critically, the representation options were the same for all of the statements. Both of the representations depicted the value of 1/2 (.50), which was not used in any of the fractions or decimals given in the statements. The choice of representation type thus could only be guided by its abstract form (continuous or discrete), rather than by matches of specific values. Participants were given eight expressions that paired number type and entity type. For each expression participants were instructed to choose which type of diagram (circle or dots) they would prefer to use to represent it.

Results

The left panel of Figure 3 shows the percentage of total times the continuous representation (circle) versus discrete representation (dots) was chosen for a given combination of entity type and number type. Collapsing across entity type, for decimal expressions participants selected the continuous representation (circle) 64% of the time, whereas for fraction

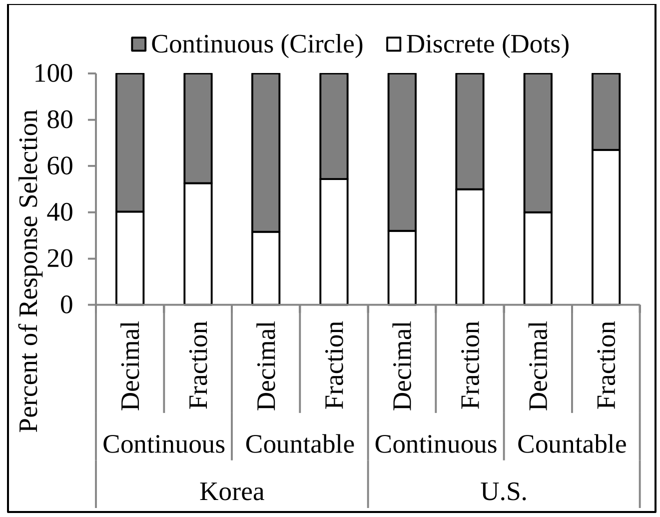

Figure 3. Percentage response selection by number type for trials with continuous entities and countable entities in Experiment 2 between South Korea (left panel) and the U.S. (right panel). The U.S. results are from Rapp et al. (2015, Ex. 2, metric units only).

expressions participants chose the continuous representation (circle) 46% of the time.

A 2 x 2 within-subjects ANOVA was performed on data coded as the proportion of trials on which the continuous representation (circle) was selected. For simplicity, we report the preference for continuous only. There was a significant main effect of number type, $F(1, 56) = 8.84$, $p = .004$, indicating that the continuous representation was selected more frequently for decimals than for fractions. There was no main effect of entity type ($F < 1$), nor any reliable interaction effect between number and entity type, $F(1, 56) = 1.55$, $p = .219$.

For comparison, the right panel of Figure 3 shows the comparable data from American students (based solely on items using metric units, to maximize compatibility with the items used in Korea.) As in Experiment 1, Korean students showed the same basic pattern of alignments as had been found for American students. However, Korean students chose continuous versus countable representations equally often when representing fractions, whereas the U.S. students chose countable representation more often than continuous representation. Korean students thus showed an overall preference for continuous representations.

Experiment 3

Experiment 2 showed that participants’ preferences for representation types varied depending on the type of rational number used. In Experiment 3 we tested for alignment in the opposite direction. College students were asked to choose either a fraction or decimal for different types of displays that depicted ratio relations. Experimental materials were adapted from Experiment 1 of DeWolf et al. (2015).

Participants

A total of 60 undergraduate students (male = 18; mean age = 21.08) from Yonsei University participated in the study for course credit. Participants were randomly assigned in equal numbers to two between-subjects conditions (part-to-part vs. part-to-whole ratio; see below).

Design, Materials, and Procedures

The study was a 2 (relation type: part-to-part vs. part-to-whole ratios) X 3 (display type: continuous, discretized, discrete) design, where relation type was a between-subjects factor, and display type was a within-subjects factor. A part-to-part ratio (PPR) is the relation between the size of the two subsets of a whole, whereas a part-to-whole ratio (PWR) is the relation between the size of one subset and the whole.

Figure 4 depicts examples of the three display types. The discrete items were displays of circles, squares, stars, crosses, trapezoids, and cloud-like shapes. The continuous items were displays of rectangles that could differ in width, height and orientation (vertical or horizontal). The discretized items were identical to the continuous displays except that the rectangles were divided into equal-sized units by dark lines. For the stimuli used in test trials, red and green were used to demarcate the two different subsets. The displays varied which color represented the larger subset versus the smaller subset.

Participants were given instructions for either the part-to-part ratio (PPR) or part-to-whole ratios (PWR) condition. They were given a Korean translation of the following instructions for the PPR condition: "In this experiment, you will see displays that show various part to part relations. In the display below [display with 1 orange circle and 2 blue crosses] this would be the number of orange circles relative to the number of blue crosses. Such relations can be represented with fractions (e.g., 3/4) or with decimals (e.g., .75). For each display your task is to choose which notation is a better representation of the depicted relation—a fraction or a decimal. Note that the specific values (i.e., 3/4 and .75) are just examples and do not match the values in the displays." For the PWR condition, the instructions were identical except for the description of the relations. In this condition the part-to-whole relation was defined using the example of the number of orange circles relative to the total number of blue crosses and orange circles. The relation type (PPR vs. PWR) was manipulated between subjects; thus participants in the PPR condition were only told about PPRs and participants in the PWR condition were only told about PWRs. Participants were shown examples of the continuous and discretized displays, in addition to the discrete display, and were told that displays could appear in any of those formats.

The task was simply to decide whether the relationship should be represented with a fraction (3/4) or a decimal (.75). In order to assess this preference on a conceptual level, the specific fraction and decimal shown to participants (3/4 and .75) were held constant across all trials, and never matched the number of items in the pictures. Thus, no

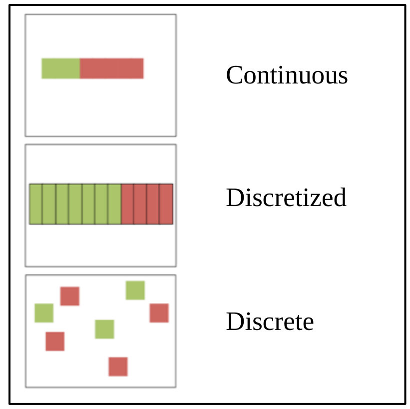

Figure 4. Examples of continuous, discretized and discrete displays used in Experiment 3.

mathematical task needed to be performed. There was therefore no requirement for accuracy, nor was any speed pressure imposed. Since the quantity shown in a display never matched the particular fraction and decimal values provided as response options, there was no real need to even determine the specific value represented in a display. The paradigm of Experiment 3 was thus intended to investigate participants' conceptual representations for fractions and decimals, in a situation in which mathematical procedures were not required.

Stimuli were displayed on a computer screen and participant responses were recorded. Participants were given the instructions described above for either the PPR condition or the PWR condition. Participants were told to select the *z* key for decimals and the *m* key for fractions. Participants completed 60 test trials (20 for each display type). A fixation cross was displayed for 600 ms between each trial. Display types were shown in a different random order for every participant. All participants were tested in a laboratory.

Results

Because participants were forced to choose either a fraction or a decimal for each trial, the preference for each is complementary. For simplicity, we report the preference for fractions. The proportion of trials in which participants selected the fraction notation was computed for each display type for each participant. The left panel of Figure 5 shows the proportion of trials that participants chose either fractions or decimals for each display. A 2 (relation type: PPR vs. PWR) X 3 (display type: discrete, discretized, continuous) ANOVA was performed to assess differences in notation preference. There was a significant main effect of display type, $F(2, 116) = 30.88$, $p < .001$. Planned comparisons showed that preference for fractions was significantly greater for discretized displays than discrete displays, $t(59) = 2.23$, $p = .029$, which in turn was greater than continuous displays, $t(59) = 4.94$, $p < .001$. There was no interaction between relation type and display type, $F(2, 116) = 1.17$, $p = .314$, and no main effect of relation type, $F < 1$.

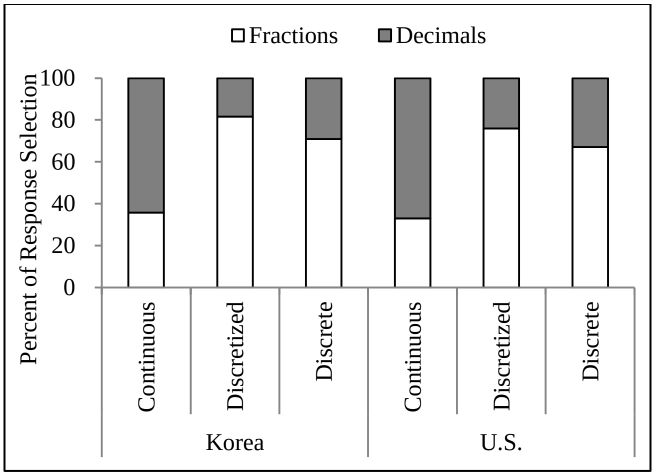

Figure 5. Percentage response selection for each display type in which either a fraction or decimal were chosen in Experiment 3 between South Korea (left panel) and the U.S. (right panel). The U.S. results are from DeWolf et al. (2015, Ex. 1).

These results reveal that Korean students preferred to represent both PPR and PWR ratio relationships with fractions when a display showed a partition of countable entities, but with decimals when the display showed a partition of continuous mass quantities. Participants picked the number format that provided the best conceptual match to either continuous or discrete displays.

No mathematical task needed to be performed, and the specific quantities depicted in the displays did not match the numerical values of the fractions and decimals provided as choice options; hence our findings demonstrate that the preferential association of display types (discrete or continuous) and rational number formats (fractions or decimals) has a conceptual basis for Korean as well as American students (DeWolf et al., 2015). This result closely aligns with the results of Experiments 1-2, in that college-educated adults show a preference for using continuous displays to represent decimals and countable displays to represent fractions. The patterns of results were consistent between Korea and the U.S. Experiment 3 thus provides strong support for the hypothesis that the natural alignment of different symbolic notations with different quantity types has a conceptual basis.

Discussion

The results across the three experiments conducted in Korea revealed a pattern very similar to that obtained using adults in the United States. Although direct statistical comparisons between the two countries are not possible because the experiments were done separately, the patterns of results across the two parallel sets of studies are in broad qualitative agreement. Decimals were typically used to represent continuous entities, whereas fractions were more likely to represent discrete than continuous entities. Continuity versus discreteness is a basic ontological distinction that affects children's understanding of integers through counting of discrete entities, and (later on) through measurement of continuous entities that have been parsed into discrete units (e.g., Gelman, 1993; Gelman, 2006; Mix, Huttenlocher & Levine, 2002a, 2002b; Nunes, Light & Mason, 1993; Rips, Bloomfield & Asmuth, 2008). The distinction between continuity and discreteness is preserved throughout the mathematical curriculum. As in the initial cases of counting and measurement, discrete concepts are always taught before their continuous counterparts (e.g., first arithmetic progressions, then linear functions).

The two symbolic notations of rational numbers, together with their respective alignments to discrete and continuous entities, are differentially suited for different reasoning tasks. DeWolf et al. (2015) found that fractions allow people to better represent bipartite relations between discrete sets than do decimals. This difference arises because fractions maintain the mapping of distinct countable sets onto the numerator and the denominator, whereas decimals obscure this mapping. At the same time, decimals afford direct mapping onto a mental number line, and therefore allow for easier magnitude assessment than do fractions (DeWolf et al., 2014; Iuculano & Butterworth, 2011).

Although the overall patterns of results in our experiments were consistent between the U.S. and South Korea, Korean students showed a general bias towards using continuous entities and representations in Experiments 1-2. One possible explanation is use of the metric measurement system in Korea. In a preliminary textbook analysis, we found a pattern qualitatively similar to that observed in Experiment 1. Unlike the U.S. (Rapp et al., 2015), Korean textbooks used continuous entities more often than discrete entities for both fractions and decimals (although the preference for continuous entities is reduced for fractions relative to decimals). The performance of college students in Experiment 1, and the correspondence between their performance and textbook examples, may reflect the early exposure of Korean students to this alignment in the textbook examples.

Despite this secondary difference between the patterns observed in the U.S. and Korea, the present study provides strong evidence that a natural alignment holds between entity type and rational numbers. This alignment cannot be attributed to the specifics of education, language, and measurement units, which differ greatly between the United States and South Korea. Given that we know students are particularly prone to misconceptions with rational numbers (Siegler et al., 2013; Ni & Zhou, 2005; Staflyidou & Vosniadou, 2004; Stigler, Givvin & Thompson, 2010), making use of this natural alignment may help students to use their knowledge of entities in the real world to bootstrap their knowledge of rational numbers. Interestingly, despite the prevalence of this alignment in textbooks across many grade levels, textbooks never actually address it explicitly. The alignment seems to be implicit, and is not explicitly taught even for adults. Teaching with this alignment in mind, and even explicitly using it, may provide a useful stepping-stone for children learning rational numbers. In addition, having students engage in tasks in which they need to actively parse a continuous representation, or conversely, sum over a discrete representation to align it with a decimal value, may provide a useful tool for bolstering understanding of the relation between the representations of entities and the rational numbers themselves.

More generally, the present study illustrates the importance of cross-national and cross-cultural research in the field of higher cognition (cf. Richland, Zur & Holyoak, 2007). It is critical to distinguish between phenomena that are specific to particular educational practices in specific contexts from those that reflect the fundamental representational capacities of the human mind. The methodological approach of identifying those aspects of cognitive performance that are the same or different across populations varying in culture, language, and educational practices is especially informative in answering these types of basic questions.

Acknowledgments

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